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OPTIMIZATION OF PROCESSES WITH DIFFERENCE ARGUMENTS

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Iu. V. KOZHEVNIKOV
(Kazan')
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Optimization of a process with difference arguments is considered. Necessary optimality conditions are obtained in the form of maximum principle. The problem is reduced to a boundary-value problem for a system of ordinary differential equations with no difference arguments. This is performed by a special transformation.

The damping of vibrations of a string is considered as an example.

1. Some important problems in mathematical physics such as the damping of one-dimensional vibrational processes (see Example) can be reduced to the following optimal problem.

For the process $x(t) = (x_1(t),\ldots, x_n(t))$ with the values $x \in X \subset E_n$ for each $t \in [0, k\, t_k]$ the process being described on the portions $[st_k, (s+1)t_k]$ $(s = 0,\ldots, k-1)$ by the equations

$$\left.\frac{dx}{dt}\right|_{st_k+\tau} = \varphi^s(\tau, z^+, z^-) \qquad (\tau \in [0, t_k];\ s = 0, \ldots, k-1) \tag{1.1}$$

with the boundary conditions

$$f_j(x_0^\circ, \ldots, x_0^{k-1}, x_k^\circ, \ldots, x_k^{k-1}) = 0 \qquad (j = 1, \ldots, q;\ q < 2nk) \tag{1.2}$$

it is required to find a control $u(t) = (u_1(t),\ldots, u_r(t))$ with the values $u \in U \subset E_r$ for each $t \in [0, kt_k]$ which minimizes the functional

$$J = f_0(x_0^\circ, \ldots, x_0^{k-1}, x_k^\circ, \ldots, x_k^{k-1}) \tag{1.3}$$

where $x(t)$ is a continuous time vector-function and $u(t)$ is a piece-wise continuous time vector-function on $[st_k, (s+1)t_k]$; $\varphi^s = (\varphi_1^s, \ldots, \varphi_n^s)$, $f_0, \ldots, f_q$ are continuous and twice continuously differentiable ; t_k is a specified value.

$$z^+ = (x(\tau),\ldots, x((k-1)\,t_k+\tau),\ u(\tau),\ldots,\ u((k-1)t_k+\tau))$$
$$z^- = (x(t_k-\tau),\ldots, x(kt_k-\tau),\ u(t_k-\tau),\ \ldots,\ u(kt_k-\tau))$$
$$x_0^s = (x_{i0}^s) = x(st_k+\tau)\big|_{\tau=0} = x((s+1)\,t_k-\tau)\big|_{\tau=t_k}$$
$$x_k^s = (x_{ik}^s) = x(st_k+\tau)\big|_{\tau=t_k} = x((s+1)\,t_k-\tau)\big|_{\tau=0}$$

The solution of the problem as formulated above is called an optimal solution.

In view of the specific character of the difference argument and of the definition domain of the solution on the time axis, it is possible to formulate such necessary conditions of optimality that enable one to apply the maximum principle of Pontriagin[1]; the problem is thus reduced to a standard boundary-value problem for an enlarged system of ordinary differential equations with no difference arguments.

We introduce new phase coordinates $x^s = x^s(\tau)$, $x^{s-} = x^{s-}(\tau)$ and new control functions $u^s = u^s(\tau)$, $u^{s-} = u^{s-}(\tau)$ related to the original ones

$$x^s(\tau) = x(st_k+\tau),\quad x^{s-}(\tau) = x((s+1)\,t_k-\tau),\quad u^s(\tau) = u(st_k+\tau) \tag{1.4}$$
$$u^{s-}(\tau) = u((s+1)\,t_k-\tau)$$
$$(s=0,\ldots,k-1)$$

Obviously

$$x^s(\tau) = x^{s-}(t_k-\tau),\qquad u^s(\tau) = u^{s-}(t_k-\tau)$$
$$x^s(0) = x^{s-}(t_k) = x_0^s,\qquad x^s(t_k) = x^{s-}(0) = x_k^s \tag{1.5}$$
$$\frac{dx}{dt}\bigg|_{st_k+\tau} = \frac{dx^s}{d\tau},\qquad \frac{dx^s}{d\tau}\bigg|_{t_k-\tau} = -\frac{dx^{s-}}{d\tau}$$

Hence and from (1.1) it follows that the vector-functions

$$x^+ = (x_1^\circ,\ldots,x_n^{k-1}),\ x^- = (x_1^{\circ -},\ldots,x_n^{(k-1)-}),\quad u^+ = (u_1^\circ,\ldots,u_r^{k-1})$$
$$u^- = (u_1^{\circ -},\ldots,u_r^{(k-1)-})$$

satisfy the equations

$$\frac{dx_i^s}{d\tau} = \varphi_i^s(\tau, x^+, x^-, u^+, u^-),\quad \frac{dx_i^{s-}}{d\tau} = -\varphi_i^{s-}(\tau, x^+, x^-, u^+, u^-) \tag{1.6}$$
$$(\tau\in[0, t_k];\ i=1,\ldots,n;\ s=0,\ldots,k-1)$$

Also

$$x_0^s = x^s(0),\quad x_k^s = x^s(t_k),\quad x_0^{s-} = x^{s-}(0),\quad x_k^{s-} = x^{s-}(t_k)$$

are related by the expression (1.2) and the conditions

$$x_0^{s-} - x_k^s = 0,\qquad x_k^{s-} - x_0^s = 0\qquad (s=0,\ldots,k-1) \tag{1.7}$$

The vector-function $\varphi^{s-} = (\varphi_1^{s-},\ldots,\ \varphi_n^{s-})$ is obtained from $\varphi^s = (\varphi_1^s,\ldots,\varphi_n^s)$ by replacing its argument τ by $t_k-\tau$ and by inverting the superscripts plus and minus of the other arguments. We set

$$H(\tau, x^+, x^-, u^+, u^-, \lambda^+, \lambda^-) = \sum_{s=0}^{k-1}[(\lambda^s, \varphi^s) + (\lambda^{s-}, \varphi^{s-})] \tag{1.8}$$

where

$$\lambda^+ = (\lambda_1^\circ,\ldots,\lambda_n^{k-1}),\quad \lambda^- = (\lambda_1^{\circ -},\ldots,\lambda_n^{(k-1)-}),\quad \lambda^+(\tau) = \lambda^-(t_k-\tau)$$

are defined from the equations

$$\frac{d\lambda_i^s}{d\tau} = -\frac{\partial H}{\partial x_i^s},\quad \frac{d\lambda_i^{s-}}{d\tau} = \frac{\partial H}{\partial x_i^{s-}}\quad \begin{pmatrix}\tau\in[0, t_k];\ i=1,\ldots,n\\ s=0,1,\ldots,k-1\end{pmatrix} \tag{1.9}$$

together with the boundary conditions

$$\lambda_{ik}^{s} = \lambda_{i0}^{s-} = -\sum_{j=0}^{q} \nu_j \frac{\partial f_j}{\partial x_{ik}^{s}}, \qquad \lambda_{i0}^{s} = \lambda_{ik}^{s-} = \sum_{j=0}^{q} \nu_j \frac{\partial f_j}{\partial x_{i0}^{s}} \tag{1.10}$$
$$(s = 0, \ldots, k-1;\ i = 1, \ldots, n)$$

Here $\nu_0 = 1, \nu_1, \ldots, \nu_q$ are the constant Lagrange multipliers, and

$$\lambda_{i0}^{s} = \lambda_{i0}^{s}(0),\ \lambda_{ik}^{s} = \lambda_i^{s}(t_k),\ \lambda_{i0}^{s-} = \lambda_i^{s-}(0),\ \lambda_{ik}^{s-} = \lambda_i^{s-}(t_k)$$

Necessary conditions for optimality of the control $u(t)$ and hence a method for solving the problem are given in the following theorem.

Theorem 1. In order that the controls $u(t)$, $t \in [0, kt_k]$ be optimal it is necessary that for each $\tau \in [0, t_k]$ the function $H(\tau, x^+, x^-, u^+, u^-, \lambda^+, \lambda^-)$ of the variables $u^s \in U$, $u^{s-} \in U$ attain its maximum on the values $u(t)$.

It follows from the theorem that in the open kernel of the region U the optimal control satisfies the equations

$$\frac{\partial H}{\partial u_j^s} = 0, \quad \frac{\partial H}{\partial u_j^{s-}} = 0 \quad (j = 1, \ldots, r;\ s = 0, \ldots, k-1) \tag{1.11}$$

It is noted that the functions x^+, x^-, u^+, u^- should be regarded as ordinary phase coordinates and control functions of the argument $\tau : \tau \in [0, t_k]$. Thus by Theorem 1 the problem is reduced to the integration of Eqs. (1.6) and (1.9) together with the necessary optimality conditions for u^+, u^- as given, for example, by (1.11) with the boundary conditions (1.2), (1.7) and (1.10), i.e. to the solution of a boundary-value problem without a difference argument.

The proof of the theorem is given in the Appendix.

2. We shall now consider the problem of damping the string vibrations which are governed by the wave equation

$$z_{tt} - a^2 z_{\xi\xi} = F(t, \xi) \quad (0 < \xi < l,\ 0 < t < 2t_k) \tag{2.1}$$

satisfying the initial and boundary conditions

$$z|_{t=0} = \varphi(\xi), \quad z_t|_{t=0} = \psi(\xi), \quad z|_{\xi=l} = 0, \quad z|_{\xi=0} = x_1(t) \tag{2.2}$$

Here $z = z(t, \xi)$ is the deviation of the string from its equilibrium position, a is the propagation velocity of excitation, $F(t, \xi)$, $\varphi(\xi)$, $\psi(\xi)$ are given functions which have the required function-theoretical properties, $x_1(t)$ is the motion of the left end of the string to be selected in such a manner that damping results, a, l, $t_k = l / a$ are known quantities.

One can adopt as a measure of damping the string its energy at the instant $t = 2t_k$

$$I = \frac{1}{2}\int_0^l (T z_t^2|_{2t_k} + \rho z_\xi^2|_{2t_k})\, d\xi \qquad (T = a^2\rho) \tag{2.3}$$

where T and ρ are tension and mass density of the string respectively [2].

The problem now consists in determining $x_1(t)$, $t \in [0, 2t_k]$, $x_1(0) = \varphi(0)$ which minimize the functional (2.3).

The solution of Eq. (2.1) satisfying the conditions (2.2) can be written in the following form [2]:

$$z(t, \xi) = z_0(t, \xi) + z_1(t, \xi)$$

Here $z_0(t, \xi)$ is the solution of Eq. (2.1) under the conditions (2.2) in which the last relation has been replaced by $z|_{\xi=0} = 0$; the functions $z_1(2t_k, \xi)$, $z_{1\xi}(2t_k, \xi)$, $z_{1t}(2t_k, \xi)$ are given by the formulas

$$z_1|_{2tk} = x_1(2t_k - \xi/a) - x_1(\xi/a), \qquad z_{1\xi}|_{2tk} = -a^{-1}[u(2t_k - \xi/a) + u(\xi/a)]$$

$$z_{1t}|_{2tk} = u(2t_k - \xi/a) - u(\xi/a), \; u(t) = dx_1/dt \tag{2.4}$$

By inserting the expressions (2.4) into the right side of (2.3) we find after simple transformation that the minimization of I leads to minimization of the functional

$$J = x_{2k}^{\circ} \tag{2.5}$$

defined by the system of equations with difference arguments as in (1.1),

$$\frac{dx_1}{dt}\Big|_{\tau} = u(\tau), \qquad \frac{dx_1}{dt}\Big|_{t_k+\tau} = u(t_k + \tau)$$
$$\frac{dx_2^{\circ}}{d\tau} = [u(\tau)]^2 + [u(2t_k - \tau)]^2 + A_1(\tau)u(2t_k - \tau) + A_2(\tau)u(\tau) \tag{2.6}$$

with the boundary conditions

$$x_{20}^{\circ} = 0, \quad x_1(\tau)|_{\tau=0} = \varphi(0), \quad x_1(\tau)|_{\tau=t_k} = x_1(t_k + \tau)|_{\tau=0} \quad (\tau = \xi/a) \tag{2.7}$$

$$A_1(\tau) = a^{-1}z_{0t}(2t_k, a\tau) - z_{0\xi}(2t_k, a\tau), \; A_2(\tau) = -a^{-1}z_{0t}(2t_k, a\tau) - z_{0\xi}(2t_k, a\tau) \tag{2.8}$$

The system (2.6) and the boundary conditions (2.7) are reduced to the following relations analogous to (1.2), (1.6) and (1.7):

$$\frac{dx_1^{\circ}}{d\tau} = u^{\circ}, \qquad \frac{dx_1^{\circ-}}{d\tau} = -u^{\circ-}$$
$$\frac{dx_1^{1}}{d\tau} = u^{1}, \qquad \frac{dx_1^{1-}}{d\tau} = -u^{1-}$$
$$\frac{dx_2^{\circ}}{d\tau} = (u^{\circ})^2 + (u^{1-})^2 + A_1u^{1-} + A_2u^{\circ}, \quad \frac{dx_2^{\circ-}}{d\tau} = -(u^{\circ-}) + (u^1)^2 - A_1(t_k - \tau)u^1 - A_2(t_k - \tau)u^{\circ-}$$
$$x_{10}^{\circ} = \varphi(0), \quad x_{1k}^{\circ} - x_{10}^{1} = 0, \quad x_{20}^{\circ} = 0, \; x_{i0}^{s-} = x_{ik}^{s}, \; x_{ik}^{s-} = x_{i0}^{s} \; (i = 1, 2; \; s = 0, 1$$

Thus in the case under consideration

$$H = \lambda_1^{\circ}u^{\circ} + \lambda_1^{1}u^{1} + \lambda_2^{\circ}[(u^{\circ})^2 + (u^{1-})^2 + A_1u^{1-} + A_2u^{\circ}] + \lambda_1^{\circ-}u^{\circ-} + \lambda_1^{1-}u^{1-} + \lambda_2^{\circ-}[(u^{\circ-})^2 + (u^1)^2 + A_1(t_k - \tau)u^1 + A_2(t_k - \tau)u^{\circ-}]$$

The expression for H does not contain the phase coordinates explicitly; therefore we have

$$\lambda_1^{\circ} = \lambda_1^{\circ-} = \text{const}, \; \lambda_1^{1} = \lambda_1^{1-} = \text{const}, \; \lambda_2^{\circ} = \lambda_2^{\circ-} = \text{const}$$

From the transversality conditions (1.10) we obtain

$$\lambda_1^{\circ} = \lambda_1^{\circ-} = \lambda_1^{1} = \lambda_1^{1-} = 0, \qquad \lambda_2^{\circ} = \lambda_2^{\circ-} = -1$$

and from the equations

$$\partial H / \partial u^{\circ} = 0, \qquad \partial H / \partial u^{1} = 0$$

we find that

$$u^{\circ}(\tau) = u(\tau) = -\tfrac{1}{2}\, A_2(\tau), \qquad u^{1}(\tau) = u(t_k + \tau) = -\tfrac{1}{2}\, A_1(t_k - \tau)$$

Consequently, the sought optimal control $x_1(t)$ is given by the expressions

$$x_1(\tau) = \varphi(0) - \frac{1}{2}\int_0^{\tau} A_2(\tau)\, d\tau \qquad (0 \leqslant \tau \leqslant t_k) \tag{2.9}$$

$$x_1(t_k+\tau) = \varphi(0) - \frac{1}{2}\int_0^{t_k} A_2(\tau)\,d\tau - \frac{1}{2}\int_0^{\tau} A_1(t_k-\tau)\,d\tau \qquad \text{(Cont.)}$$

In the particular case of

$$l = t_k = \pi, \quad a = 1, \quad \varphi(\xi) = A\sin\xi, \quad \psi(\xi) = F(t,\xi) = 0$$

we have

$$z_0|_{2t_k} = A\sin\xi, \quad z_{0t}|_{2t_k} = 0, \quad z_{0\xi}|_{2t_k} = A\cos\xi, \quad A_1(\tau) = A_2(\tau) = -A\cos\tau$$

Consequently,

$$x_1(\tau) = {}^1/_2\, A\, \sin\tau, \qquad x_1(t_k+\tau) = -{}^1/_2\, A\, \sin\tau$$

Together the above formulas yield

$$x_1(t) = {}^1/_2 A\sin t \qquad (0 \leqslant t \leqslant 2t_k) \tag{2.10}$$

Then, in view of (2.4) we find

$$z_1|_{2t_k} = -A\sin\xi, \qquad z_{1t}|_{2t_k} = 0$$

and we have now

$$z|_{2t_k} = z_0|_{2t_k} + z_1|_{2t_k} = 0, \qquad z_t|_{2t_k} = z_{0t}|_{2t_k} + z_{1t}|_{2t_k} = 0$$

Thus, the control (2.10) brings the string into an equilibrium position.

Appendix. It is obvious that having introduced the vector-functions x^+, x^-, u^+, u^- the optimization of $u(t)$ reduces to the solving of the following optimal problem: for the system (1.6) with phase state x^+, x^- satisfying the boundary conditions (1.2) and (1.7) it is required to find the control u^s, u^{s-} with values in U which minimizes the functional (1.3). In view of (1.5) the functions u^+ and u^- are connected by the relations

$$u^+(\tau) = u^-(t_k-\tau), \qquad u^-(\tau) = u^+(t_k-\tau) \tag{A.1}$$

We note that (1.6), (1.7) and (A.1) imply

$$x^+(\tau) = x\ (t_k-\tau), \qquad x^-(\tau) = x^+(t_k-\tau) \tag{A.2}$$

This indicates that the conditions (1.5) on x^s, x^{s-} have become identities.

It is obvious that to solve the above formulated problem one can apply the maximum principle of Pontriagin. The construction of almost impulsive variation of the control should be considered independently. This special feature related to the condition (A.1) implies the following: if on an arbitrary small interval $[\tau', \tau''] \subset [0, t_k]$ the control $u^+(\tau)$ is associated with an almost impulsive variation, that is, if $u^+(\tau) = \omega, \omega \in U$, then we have simultaneously $u^-(\tau) = \omega$, $\tau \in [t_k - \tau'', t_k - \tau']$. This results in an independent nonpositive term appearing in the expression for the linear part of the increment of the functional J

$$\int_{\tau'}^{\tau''} [H^*(\tau, x^+, x^-, \omega, u^-, \mu^+, \mu^-) - H^*(\tau, x^+, x^-, u^+, u^-, \mu^+, \mu^-)]\,d\tau +$$

$$+\int_{t_k-\tau''}^{t_k-\tau'} [H^*(\tau, x^+, x^-, u^+, \omega, \mu^+, \mu^-) - H^*(\tau, x^+, x^-, u^+, u^-, \mu^+, \mu^-)]\,d\tau \leqslant 0 \tag{A.3}$$

$$H^* = \sum_{s=0}^{k-1} [(\mu^s, \varphi^s) - (\mu^{s-}, \varphi^{s-})]$$

where x^+, x^-, u^+, u^- define the optimal state, and ω corresponds to the control almost impulsive variation; $\mu^+ = (\mu_1^\circ, \ldots, \mu_n^{k-1})$, $\mu^- = (\mu_1^{\circ-}, \ldots \mu_n^{(k-1)-})$ are adjoint functions which are defined from the equations [3]

$$\frac{d\mu_i^s}{d\tau} = -\frac{\partial H^*}{\partial x_i^s}, \qquad \frac{d\mu_i^{s-}}{d\tau} = -\frac{\partial H^*}{\partial x_i^{s-}} \begin{pmatrix} \tau \in [0, t_k]; i = 1, \ldots, n \\ s = 0, 1, \ldots, k-1 \end{pmatrix} \tag{A.4}$$

with the boundary conditions

$$\mu_{ik}^s = \nu_i^{s0} - \sum_{j=0}^{q} \nu_j \frac{\partial f_j}{\partial x_{ik}^s}, \quad \mu_{i0}^s = -\nu_i^{sk} + \sum_{j=0}^{q} \nu_j \frac{\partial f_j}{\partial x_{i0}^s}, \mu_{ik}^{s-} = -\nu_i^{sk}, \mu_{i0}^{s-} = \nu_i^{s0}$$

$$(i = 1, \ldots, n;\ s = 0, \ldots, k-1) \tag{A.5}$$

In the relations (A. 5) the subscripts zero and k of μ correspond to its values at $\tau = 0$ and $\tau = t_k$; ν_i^{s0}, ν_i^{sk} are constant Lagrange multipliers.

It can be seen that in addition to the usual first term in the maximum principle theorem the left side of (A. 3) contains also another term. We change the variables under the integral sign in the second integral of (A. 3) by $\tau = t_k - t$ subsequently we return to the previous notation for the independent variable and we obtain

$$\int_{\tau'}^{\tau''} [H^* (\tau, x^+, x^-, \omega, u^-, \mu^+, \mu^-) + H^* (\tau, x^+, x^-, u^+, \omega, \mu^+, \mu^-) \mid_{t_k - \tau} -$$
$$- H^*(\tau, x^+, x^-, u^+, u^-, \mu^+, \mu^-) - H^*(\tau, x^+, x^-, u^+, u^-, \mu^+, \mu^-) \mid_{t_k-\tau}]\, d\tau \leqslant 0$$

By writing the integrand and using obvious relations

$$\varphi^s \mid_{t_k-\tau} = \varphi^{s-}, \quad \varphi^{s-} \mid_{t_k-\tau} = \varphi^s$$

as well as the relations

$$\lambda^s = \mu^s - \mu^{s-} \mid_{t_k-\tau}, \quad \lambda^{s-} = \mu^s \mid_{t_k-\tau} - \mu^{s-} \qquad (s = 0, \ldots, k-1) \tag{A.6}$$

whose validity is shown below, we obtain

$$\int_{\tau'}^{\tau''} [H (\tau, x^+, x^-, \omega, u^-, \lambda^+, \lambda^-) - H (\tau, x^+, x^-, u^+, u^-, \lambda^+, \lambda^-)]\, d\tau \leqslant 0$$

Then by employing the usual considerations of the maximum principle theory we establish the validity of the assertion of Theorem 1 as regards u^{s-}. Similarly, by associating on $[\tau', \tau''] \subset [0, t_k]$ an almost impulsive variation with the control u^{s-} we prove the assertion of the theorem as regards u^{s-}. It remains to show that λ^s, λ^{s-} introduced in (A. 6) satisfy the equality $\lambda^s|_{t_k-\tau} = \lambda^{s-}$, and the Eqs. (1. 9) as well as the boundary conditions (1. 10). The relation $\lambda^s|_{t_k-\tau} = \lambda^{s-}$ follows directly from the comparison of the right sides in (A. 6).

The conditions (A. 5) can be written as

$$\mu_{ik}^s - \mu_{i0}^{s-} = -\sum_{j=0}^{q} \nu_j \frac{\partial f_j}{\partial x_{ik}^s}, \qquad \mu_{i0}^s - \mu_{ik}^{s-} = \sum_{j=0}^{q} \nu_j \frac{\partial f_j}{\partial x_{i0}^s}$$

It follows directly that the left sides in (A. 3) satisfy the relations (1. 10)

Since

$$\frac{d\mu^s}{d\tau}\Big|_{t_k-\tau} = -\frac{d\mu^s (t_k - \tau)}{d\tau}, \quad \frac{d\mu^{s-}}{d\tau}\Big|_{t_k-\tau} = -\frac{d\mu^{s-} (t_k - \tau)}{d\tau}$$

therefore

$$\frac{d\lambda^s}{d\tau} = \frac{d\mu^s}{d\tau} + \frac{d\mu^{s-}}{d\tau}\Big|_{t_k-\tau}, \quad \frac{d\lambda^{s-}}{d\tau} = -\frac{d\mu^s}{d\tau}\Big|_{t_k-\tau} - \frac{d\mu^{s-}}{d\tau} \tag{A.7}$$

The relations (A.4) imply that

$$\left.\frac{d\mu_i^s}{d\tau}+\frac{d\mu_i^{s-}}{d\tau}\right|_{t_k-\tau}=-\frac{\partial H^*}{\partial x_i^s}-\left.\frac{\partial H^*}{\partial x_i^{s-}}\right|_{t_k-\tau}$$

$$\left.\frac{d\mu_i^s}{d\tau}\right|_{t_k-\tau}+\frac{d\mu_i^{s-}}{d\tau}=-\left.\frac{\partial H^*}{\partial x_i^s}\right|_{t_k-\tau}-\frac{\partial H^*}{\partial x_i^{s-}}$$

By writing out the right sides, cancelling identical terms and using (A.6) and (A.7) and the obvious relations

$$\left.\frac{\partial\varphi_j^{\nu-}}{\partial x_i^{s-}}\right|_{t_k-\tau}=\frac{\partial\varphi_j^{\nu}}{\partial x_i^{s}}, \qquad \left.\frac{\partial\varphi_j^{\nu}}{\partial x_i^{s-}}\right|_{t_k-\tau}=\frac{\partial\varphi_j^{\nu-}}{\partial x_i^{s}}$$

we find that the left sides of (A.6) also satisfy the system (1.9).

The theorem has been proved in its entirety.

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BOUNDARY LAYER AND ITS INTERACTION WITH THE INTERIOR STATE OF STRESS OF AN ELASTIC THIN SHELL

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A.L. GOL'DENVEIZER
(Moscow)

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The theory of the interior state of stress constructed in [1] in conformity with the scheme described in [2] is supplemented by an asymptotic boundary layer theory (a theory of edge states of stress) and the question of boundary layer interaction with the interior state of stress is solved for a thin elastic isotropic shell.

A two-dimensional linear theory of thin elastic shells is formulated at the end. It is based on the results herein and in [1], and is an extension of the classical theory of shells in the sense that it permits a more exact construction of the interior state of stress and, in a certain approximation, the investigation of edge elastic phenomena not taken into account by classical theory. The interior state of stress is computed by the method proposed by using equations and boundary conditions of classical theory, which are insignificantly modified, and the computations of the edge stresses reduces to the construction of a linear combination of solutions of certain auxiliary plane and antiplane problems with standard conditions independent of the geometric properties of the shell and of the nature of its loading.